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CONSTANT LINE SOURCES OF HEAT IN INFINITE MEDIA, WHOSE THERMAL RESISTIVITIES ARE LINEAR FUNCTIONS OF THE TEMPERATURE

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Abstract— A mathematical expression is derived, which gives the temperature ϑ in an infinite medium as an *exact* function of the time τ and of the distance ρ from an infinitely long line source of constant strength φ_0, on the assumption that the thermal resistivity $1/\lambda$ of the medium is a linear function, $1/\lambda = (1 + \alpha\vartheta)/\lambda_0$, of the temperature.

Diagrams illustrate numerically the solution of the problem. They show some interesting and perhaps unexpected features of ϑ as a function of τ and ρ.

The results obtained by the formula given here may be used to improve the theories of unsteady hot wire methods for determining thermal conductivities of liquids by taking into account the effect of the temperature dependence of the thermal conductivity.

NOMENCLATURE

Dimensional quantities, Greek symbols

- α, temperature coefficient of the thermal resistivity;
- ϑ, temperature of the medium as a function of τ and ρ;
- $\varkappa, \varkappa_0$, thermal diffusivity of the medium at the temperature $\vartheta = \vartheta$ and $\vartheta = 0$ resp.;
- λ, λ_0, thermal conductivity of the medium at the temperature $\vartheta = \vartheta$ and $\vartheta = 0$ resp.;
- $\lambda/\varkappa$, heat capacity per unit volume of the medium supposed to be constant, thus equal to $\lambda_0/\varkappa_0$;
- ρ, distance from the line source;
- τ, time counted from the start of the heating;
- φ, flux of heat per unit length through a cylindrical surface at the distance ρ from the line source;
- φ_0, constant flux of heat from the line source.

Dimensionless numbers

- A, dimensionless temperature coefficient of the thermal resistivity of the medium;
- S, dimensionless flux of heat;
- T, dimensionless temperature increase;
- Z, dimensionless time (Fourier number).

INTRODUCTION

WE SUPPOSE heat to be generated in a straight line of infinite length in an infinite, homogeneous and isotropic medium, at a constant rate φ_0 per unit time and per unit length of the line, the supplying of heat starting at the time $\tau = 0$, the medium at the moment being everywhere at zero temperature in the temperature scale chosen.

Our task is to calculate exactly the temperature and express it as a function of the time τ and of the distance ρ from the line source if it is supposed that the thermal resistivity of the medium, $1/\lambda$, is a linear function of its temperature, viz.*

$$1/\lambda = (1/\lambda_0)(1 + \alpha\vartheta)$$

As regards the other properties of the medium it may be supposed that its density and its specific heat both are constant so that $\lambda/\varkappa = \text{const} = \lambda_0/\varkappa_0$.

The purpose of this investigation was primarily to find out how temperature dependent thermal resistivities of media influence the theory for unsteady hot wire methods for determining the thermal conductivities of liquids.

METHOD OF SOLUTION

We may start from the following two differential equations, written in dimensional form

$$\varphi = -2\pi\rho\lambda\frac{\partial\vartheta}{\partial\rho} \tag{1}$$

$$-\frac{\partial\varphi}{\partial\rho} = 2\pi\rho\frac{\lambda_0}{\varkappa_0}\frac{\partial\vartheta}{\partial\tau}. \tag{2}$$

The first equation being Fourier's equation for the flux of heat through a cylindrical surface at a distance ρ from the line source. The latter equation expresses the absence of heat sources in the space $\rho > 0$.

Introduction of the following dimensionless variables

$$T = \frac{\lambda_0\vartheta}{\varphi_0} \qquad Z = \frac{\varkappa_0\tau}{\rho^2} \qquad S = \frac{\varphi}{\varphi_0}$$

$$A = \frac{\alpha\varphi_0}{\lambda_0} = \text{const} \tag{3}$$

transforms equations (1) and (2) into

$$S = \frac{4\pi Z}{1 + AT}\frac{\mathrm{d}T}{\mathrm{d}Z} \tag{4}$$

$$Z = \pi\frac{\mathrm{d}T}{\mathrm{d}S}. \tag{5}$$

* The reason for choosing to study cases where the thermal *resistivities* $1/\lambda$ of the media are supposed to be linear functions of the temperature, instead of cases where the thermal *conductivities* λ are supposed to be linear functions of the temperature, is that the solution seems to be simpler in the first mentioned case.

The variables τ and ρ occur only in the expression for Z, which is possible because of the special boundary and initial conditions in our problem. Due to this there are only ordinary derivatives in equations (4) and (5). Eliminating Z from equations (4) and (5) and introducing a new auxiliary variable

$$Y = \frac{A}{1 + AT}\frac{\mathrm{d}T}{\mathrm{d}S} \tag{6}$$

we obtain

$$\frac{\mathrm{d}Y}{\mathrm{d}S} = \left(\frac{4\pi}{AS} - 1\right)Y^2. \tag{7}$$

Integration of this differential equation with separable variables gives

$$\frac{1}{Y} = -\frac{4\pi}{A}\ln S + S + C_1. \tag{8}$$

Inserting Y from equation (6) and integrating we get

$$\ln(1 + AT) = \int_0^S \frac{\mathrm{d}s}{-\frac{4\pi}{A}\ln s + s + C_1} + C_2. \tag{9}$$

We have now got T as a function of the parameter S. By combination of equations (5) and (9) we obtain Z as a function of the same parameter S.

$$Z = \frac{\pi}{A}\frac{\exp\left(\int_0^S \frac{\mathrm{d}s}{-\frac{4\pi}{A}\ln s + s + C_1} + C_2\right)}{-\frac{4\pi}{A}\ln S + S + C_1}. \tag{10}$$

Boundary conditions and determination of the constants C_1 *and* C_2

The first boundary condition we get from the assumption that the initial temperature of the medium $\vartheta = 0$ or in dimensionless variables

$$T = 0 \quad \text{for} \quad Z = 0. \tag{11}$$

Another boundary condition is obtained from the heat flux from the line source:

$$\lim_{Z\to\infty} S = 1. \tag{12}$$

By use of condition (11) it is easily seen that

$$C_2 = 0.$$

The determination of the constant C_1 by use of the boundary condition (12) is much more difficult. The difficulty arises from the fact that the denominator of the integrand in equation (9), depending on C_1, may pass zero in the interval of integration $0 \leqslant S \leqslant 1$. Because the denominator for $A > 4\pi$ has a minimum in this interval we have to consider two different cases, $A \leqslant 4\pi$ and $A > 4\pi$.

A thorough mathematical investigation shows that

$$C_1 = -1 \quad \text{for} \quad A \leqslant 4\pi$$

and

$$C_1 = \frac{4\pi}{A}\left(\ln\frac{4\pi}{A} - 1\right) \quad \text{for } A > 4\pi.$$

We thus get the solution of the problem:

$$\ln(1 + AT) = \int\limits_0^S \frac{\mathrm{d}s}{-\frac{4\pi}{A}\ln s + s - 1}$$

$$\text{for } A \leqslant 4\pi \tag{13}$$

$$Z = \frac{\pi}{A}\,\frac{\exp\left(\int\limits_0^S \frac{\mathrm{d}s}{-\frac{4\pi}{A}\ln s + s - 1}\right)}{-\frac{4\pi}{A}\ln S + S - 1}$$

and

$$\ln(1 + AT) = \int\limits_0^{S'} \frac{\mathrm{d}s}{-\ln s + s - 1}$$

$$Z = \frac{1}{4}\,\frac{\exp\left(\int\limits_0^{S'} \frac{\mathrm{d}s}{-\ln s + s - 1}\right)}{-\ln S' + S' - 1} \quad \text{for } A > 4\pi \tag{14}$$

$$S' = \frac{AS}{4\pi}.$$

From (14) it is seen that values for the case $A > 4\pi$ can be calculated from values for $A = 4\pi$. We can also see that the limit $A = 4\pi$ physically expresses a kind of saturation in the temperature-field around the line source.

For $A = 0$ we get from equations (13)

$$\lim_{A\to 0} T = -\frac{1}{4\pi}\int\limits_0^S \frac{\mathrm{d}s}{\ln s}$$

$$\lim_{A\to 0} Z = -\frac{1}{4\ln S}. \tag{15}$$

From these two equations the parameter S can be eliminated and we get

$$T = -\frac{1}{4\pi}Ei\left(-\frac{1}{4Z}\right)$$

which is the well known solution to the constant line source problem with temperature independent thermal conductivity. Different approximations can be obtained from equation (13) by expansion in series and elimination of S. For example in the interval $0 < A < 4\pi$ we get for high Z-values

$$\ln T \approx \text{const} + \frac{A}{4\pi}\ln Z$$

or

$$\ln\frac{\vartheta_2}{\vartheta_1} \approx \frac{A}{4\pi}\ln\frac{\tau_2}{\tau_1}$$

which may be used for determination of A from unsteady hot-wire measurements.

NUMERICAL RESULTS

On the basis of equation (13) numerical values for T as a function of Z has been calculated for different values of A. The result is graphically shown in the Figs. 1 and 2. From these figures it can be seen that for small Z-values the fastest increase in temperature is achieved if A is negative but for higher Z-values, positive values of A give the fastest increase in temperature. For $A < 0$ the curve has a horizontal asymptote $T = -1/A$ for high Z-values.

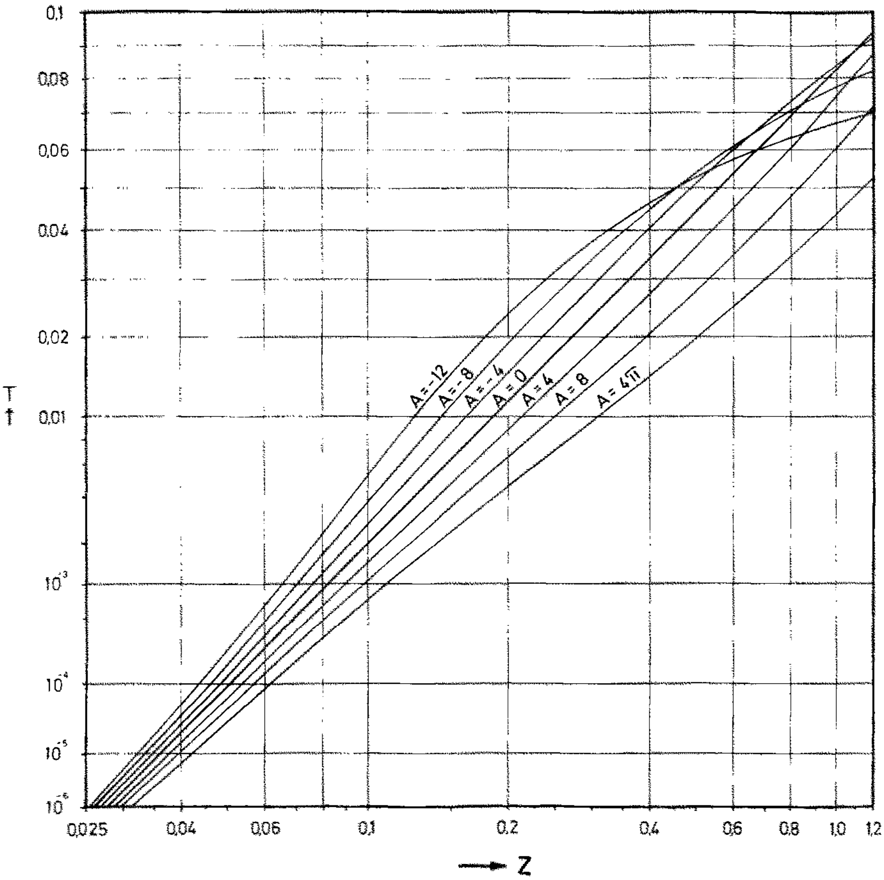

FIG. 1. T as a function of Z for small Z-values.

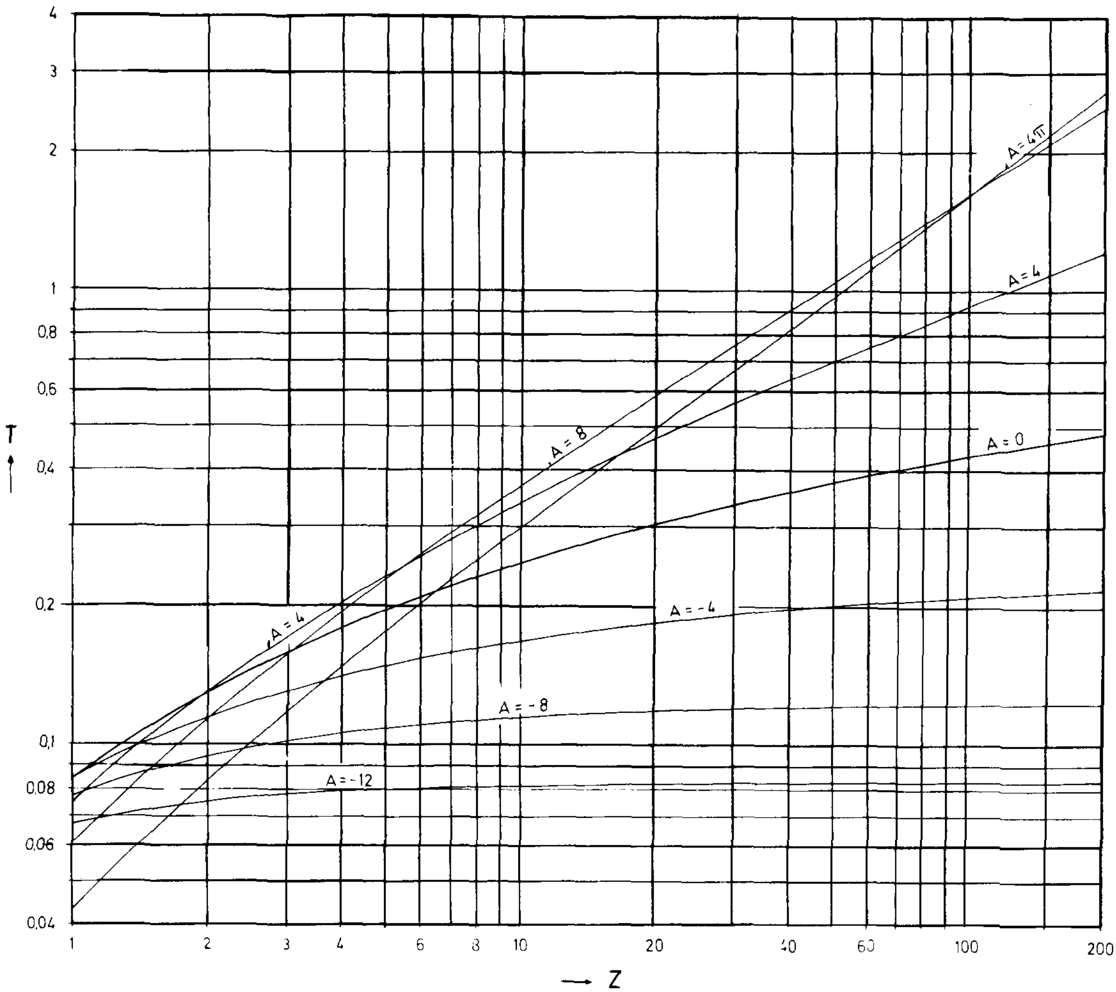

FIG. 2. T as a function of Z for high Z-values.

SOURCES DE CHALEUR LINEAIRES CONSTANTES DANS UN MILIEU INFINI A RESISTIVITE THERMIQUE FONCTION LINEAIRE DE LA TEMPERATURE

Résumé—On a obtenu une expression mathématique qui donne la température ϑ dans un milieu infini comme une fonction *exacte* du temps τ et de la distance ρ à une source linéaire infiniment longue à puissance φ_0 constante, en supposant que la résistivité thermique $1/\lambda$ du milieu est une fonction linéaire de la température $1/\lambda = (1 + \alpha\vartheta)/\lambda_0$.

Des diagrammes illustrent numériquement la solution du problème. Ils montrent quelques aspects intéressants et peut être inattendus de ϑ comme fonction de τ et ρ.

Les résultats obtenus par la formule donnée ici peuvent être utilisés pour améliorer les théories sur les méthodes instationnaires de la détermination au fil chaud des conductivités thermiques de liquides en tenant compte de l'effet de la dépendance de la conductivité thermique vis-à-vis de la température.

TEMPERATURVERTEILUNG IN UNENDLICH AUSGEDEHNTEN MEDIEN MIT KONSTANTEN LINIENFÖRMIGEN WÄRMEQUELLEN, DEREN THERMISCHE WIDERSTÄNDE LINEARE FUNKTIONEN DER TEMPERATUR SIND

Zusammenfassung—Ein mathematischer Ausdruck wurde abgeleitet, der die Temperatur ϑ in einem unendlich ausgedehnten Medium als eine exakte Funktion der Zeit τ und des Abstandes ρ von einer unendlich langen linienförmigen Quelle konstanter Ergiebigkeit φ_0, wiedergibt, unter der Voraussetzung, dass der spezifische thermische Widerstand $1/\lambda$ des Mediums eine lineare Funktion der Temperatur ist, nämlich $1/\lambda = (1 + \alpha\vartheta)/\lambda_0$. Die numerische Lösung des Problems wird durch Diagramme veranschaulicht. Sie zeigen einige interessante und vielleicht unerwartete Besonderheiten von ϑ als Funktion von τ und ρ.

Die aus den hier angegebenen Formeln gewonnenen Ergebnisse können zur Verbesserung der Theorien für instationäre Hitzdrahtmethoden zur Bestimmung der Wärmeleitfähigkeiten von Flüssigkeiten, durch Berücksichtigung des Einflusses der Temperaturabhängigkeit der Wärmeleitfähigkeit benutzt werden.

ПОСТОЯННЫЕ ЛИНЕЙНЫЕ ИСТОЧНИКИ ТЕПЛА В БЕСКОНЕЧНЫХ СРЕДАХ, УДЕЛЬНОЕ ТЕРМИЧЕСКОЕ СОПРОТИВЛЕНИЕ КОТОРЫХ ЕСТЬ ЛИНЕЙНАЯ ФУНКЦИЯ ТЕМПЕРАТУРЫ

Аннотация—Получено математическое выражение, описывающее температуру ν в бесконечной среде как точную функцию времени τ и расстояния ρ от бесконечно длинного линейного источника постоянной мощности ϕ_0 при условии, что удельное термическое сопротивление среды $1/\lambda$ есть линейная функция температуры $1/\lambda = (1+\vartheta)/\lambda_0$.

На графиках численно представлено решение задачи. Из них видны интересные и, вероятно, неожиданные свойства ϑ как функции τ и ρ.

Полученные из приведенной формулы результаты могут быть использованы при дальнейшей разработке теорий нестационарных термоанемометрических методов для определения коэффициентов теплопроводности жидкостей с учётом влияния температурной зависимости коэффициента теплопроводности.